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Fraction Flags:

Learning from Children to Help Children Learn

TANYA AND ELLEN HAVE MADE A "FRACTION flag" (**fig. 1**) by taking a half-sheet of paper and arranging smaller pieces on top of it. They have asked themselves how much of the whole sheet of paper is covered.

Students used several strategies to present their results

Ellen: The edge parts are easy—that's just two-sixths [of a whole sheet]—but the middle part is hard.

Tanya: That's because it's a twenty-fourth on top of a twenty-fourth.

Ellen: I can see the twenty-fourth in the middle, but I don't get the two little pieces on its sides.

Tanya: [Sliding over the top twenty-fourth piece] Oh, I get it. Those two side parts make a half of a twenty-fourth together, and that's a forty-eighth.

Ellen: Okay! So the total covered on the flag is two-sixths plus one twenty-fourth plus half a twenty-fourth.

Tanya: Right! So that's four, eight, nine-and-a-half twenty-fourths! What's that in forty-eighths?

Tanya and Ellen were adding fractions, but they were not applying a memorized algorithm to a routine textbook question. These middle school students had designed their own flag and were using their informal understandings to solve a problem that they had set for themselves, doing this "addition question" by extending and applying their knowledge of how fraction quantities work.

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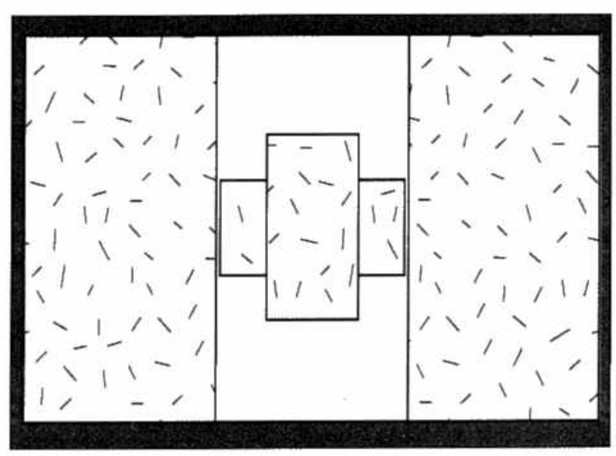


Fig. 1 Tanya and Ellen's fraction flag

This article describes "fraction flags," an activity through which fraction concepts can be explored. "Fraction flags" was actually invented by two of Tanya and Ellen's twelve-year-old classmates. Throughout this article, we placed the voices and actions of these students in the foreground, attempting to illustrate what they showed us about fractions and about learning mathematics. We begin with a brief overview of the fractions unit, outline the events that led up to the activity, and present a series of vignettes of students exploring the mathematics of "fraction flags." Through these vignettes, we describe our perceptions of the emergence of students' quantity-based personal images of fractions, their use of those images in making and solving "problems," and their contributions to the development of the curriculum of the fractions unit.

Background

THE SIX-WEEK UNIT FROM WHICH THE FLAGS activity arose was developed around various paper-manipulating activities: folding, cutting, comparing, rearranging, and assembling. As will be illustrated by the vignettes, these activities involved the physical manipulation of units and fractional subunits built from paper; mental actions on the images of fractions constructed by students; and the verbal and symbolic expression of actions, observations, and justifications. Throughout the unit, our emphasis was on presenting opportunities for students to develop a repertoire of "fraction-ing" experiences from which various mathematics concepts could emerge.

With regard to mathematics content, our guide for the unit was the mandated program of studies of our province (Alberta Education 1988). At the grade-7 level, this program emphasizes an extended informal study of operations on fractions. Most students in the class that we taught had been introduced to formal algorithms for equivalence and addition concepts in previous years, but their understandings of these concepts tended to be limited and disjoint. We should emphasize before continuing that the "knowledge objectives" of the formal curriculum did not serve as the end-points for our teaching but as boundary markers for the territory to be explored. As such, we believe that the activities that follow not only support the curriculum objectives but also offer a means of extending them.

The fractions unit centered on providing occasions for students to build their own ideas of fractions. In particular, we hoped to prompt learners to view fractions as representing additive quantities and as showing multiplicative relationships. One setting for investigating multiplicative notions involved folding units—standard pieces of paper—and subunits into various numbers of parts. For example, starting with a unit folded in half, students could explore and represent the possible foldings that would generate thirty-sixths.

Another setting, the "pizza fractions kit," offered the opportunity for students to develop an additive, quantitative sense of rational numbers. It also formed the action basis for "fraction flags."

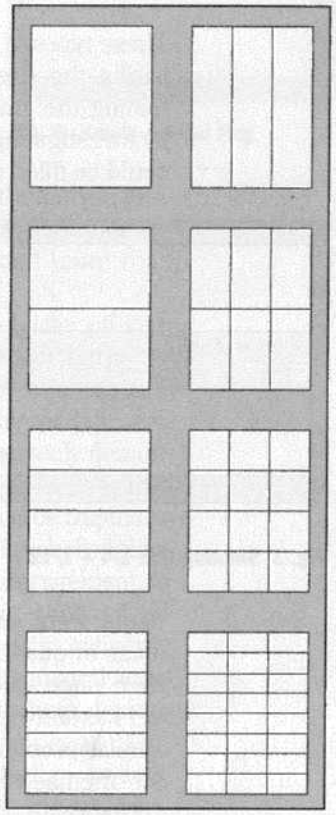


Fig. 2 The "pizza fractions" kit

"The pizza fraction kit" consisted of assorted fraction pieces—including wholes (or ones), halves, thirds, fourths, sixths, eighths, twelfths, and twenty-fourths (see **fig. 2**). To facilitate their use, pieces were color coded and stored in a large envelope. Because they were made from standard paper, the kits were inexpensive and expandable and could be made by students.

The pizza kits proved to be powerful learning tools because of an unanticipated feature of the pieces. The length-to-width ratio of standard paper is almost 4 to 3; therefore, the width of a one-third piece cut lengthwise from the paper was almost the same as the width of a one-fourth piece cut widthwise. At first, this feature was seen as problematic, but it soon became clear that it facilitated experimenting with various additive and comparative relationships. Students could quickly demonstrate, for example, that a fourth and a twelfth together covered the same area as a third (see **fig. 3**).

Working in groups of two or three, the students used their kits in various tasks, including generating "pizza orders," such as

$$\frac{3}{4}+\frac{2}{3}+\frac{5}{12}.$$

These types of "orders" could be discussed by comparing the sizes of the constituent parts, by determining the amount of pizza needed to fill them, or by investigating the different ways in which an order could be filled by using specified pieces from the kit. Thus, a number of rational-number concepts were immediately accessible in these problem-solving activities, including comparison, quantities greater than and smaller than 1, addition, common denominators, multiplication, and division. As demonstrated by their abilities to use complex fraction phrases and sentences when discussing their actions, students were able to use their already formulated ideas of quantity to build rich understandings. Further, once students had an image of rational numbers as showing amounts, they were able to work on similar problems without using the kits, generating for themselves various numerical procedures for comparison, addition, equivalence, and division. Fraction concepts were thus extended beyond the scope of the kits. Although these invented procedures were very similar or identical to the standard fraction algorithms, they served students as personal summaries of understandings based on their own actions and images.

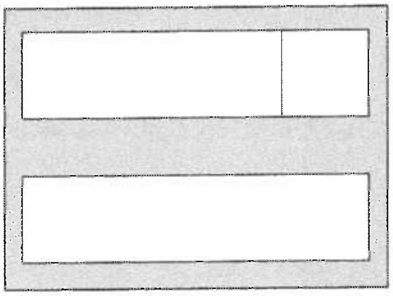


Fig. 3 Showing that 1/4 + 1/12 = 1/3

The teacher provided a setting within which the development of images and understandings were supported by students' actions and interactions. Throughout the activity, the teacher moved about the room, listening in, encouraging, questioning, offering suggestions, and requesting that emerging ideas be displayed or defended at the chalkboard. Several strategies were used by students to present their results, including drawing pictures, writing out fraction phrases and sentences, and reproducing summary charts. In most instances, all students were still actively engaged when the teacher called for a class discussion and justification of the examples on the chalkboard. This kind of open learning environment brings to mind suggestions made in the *Professional Standards for Teaching Mathematics* (NCTM 1991) regarding the role of the teacher. These discussions and interactions allowed the teacher to observe and respect the diversity in students' fractional thinking and in their expression. They also permitted the teacher to introduce and encourage the use of standard fraction language. Students were able to contribute to their own curriculum not only through the choices of actions they made within the settings or discussions but also through the natural exploration or "play" that took them beyond the tasks set by the teacher.

Fraction Flags

ONE DAY, AFTER THEY HAD DEMONSTRATED that they could work with addition of fractions, two students were playing with some pieces from the pizza kit while waiting for the teacher. Kurt placed three eighths pieces onto a half piece, as shown in **figure 4.** Luckily the teacher was within earshot.

Sandra: That's neat—like a flag! I wonder how much is left blue? [i.e., How much of the whole sheet is left uncovered?]

Kurt: [Puzzled, and not thinking about fraction amounts] What? Oh, I don't know.

Sandra: I see—it's easy! There are three eighths pieces so that leaves an eighth. So each skinny strip is a sixteenth. Two-sixteenths.

Kurt: Oh yeah.

On hearing this discussion, the teacher recognized that such "fraction flags" might hold some potential as a new setting for fraction problems. Sandra had developed a natural question using subtraction of fraction quantities and was able to "see" the amounts in this situation as mathematically derived quantities rather than in terms of known shapes. The teacher thought that flags would present an opportunity for other students to do likewise. The "fraction flags" activity was soon introduced to other students through the questions illustrated in **figure 5.**

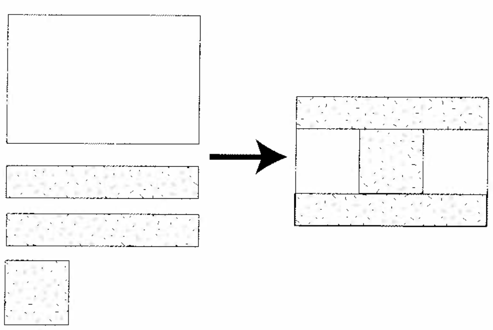
An understanding of the role of the teacher in this sort of communicative classroom setting is crucial. As might be interpreted from this article's title, the fractions unit was based on the premise that an important part of the teacher's task is *attending* to learners—that is, learning from them—to structure responsively a more effective learning environment. Mathematics teaching, so conceived, consists not of *telling* students what to do and how to do it but of providing

Fig. 4 Kurt's flag

occasions for their actions. To use a "path" metaphor, the route for students is not prespecified by the teacher but is determined interactively—according to insights, interests, and happenstance—while walking together.

Of course, we do not mean to suggest that the teacher should rely on serendipitous events while abandoning a leadership role and ignoring programs of study. Quite the contrary. Our point is that if we simply define our task as teachers in terms of having students respond in predicted ways to specified curriculum tasks, we run the risk of becoming mere *tellers*, oblivious to the richness of students' experience and the generative potential of their action. However, if the program of studies is seen as marking the "milestones" on the pathways of students' learning, we can forgo attempts to orchestrate specific responses by students and move toward offering experiences that can foster broader and deeper personal understandings.

Our point is well illustrated by the dialogue that opened this article. Ellen and Tanya had generated their flag and the related questions in response to the teacher's query about the kinds of questions their classmates would like. The girls solved the problem



Take a half piece, a twelfth piece, and two eighth pieces from your kit and make this flag. Show it to your partner, and make up some fraction questions about the flag. For example, is more of the base covered or uncovered? How much more?

Use pieces from your kit to make up your own flag. Once you have done this, make up some fraction questions about it. Try to have your partners or other students in class answer your questions. All members of your group should be ready to discuss your flags and questions with the rest of the class. ***Remember, try to make interesting flags, but also make flags so that you can ask good fraction questions about them.***

Fig. 5 "Fraction flag" questions

of how much was covered by rearranging the pieces so that they could "see" the amounts involved. Although the arithmetic was informal, it amounted to

$$\frac{2}{6}+\frac{1}{24}+\frac{1}{2}\left(\frac{1}{24}\right)=\frac{9.5}{24}=\frac{n}{48}.$$

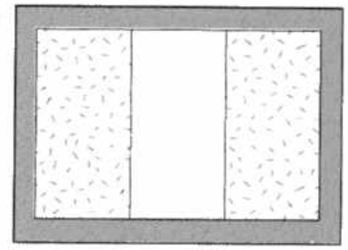


Fig. 6 Katy's second flag

Notice that these students, considered average achievers in mathematics, generated and easily worked with rational-number expressions involving many fractions and more than one operation.

Asked to describe the question, Ellen later offered, "It's good. It's a little challenging. It's not boring to do." Statements like these suggest that she was able to evaluate appropriately the difficulty of her own problems. She, and other students, repeatedly demonstrated that left to design their own questions within contexts that were of interest, they could be trusted to develop situations that were personally challenging. Further, students hardly needed encouragement to do so. The question "I'm done. What should I do now?" was simply not heard during the flags activity.

Of course, the teacher enhanced this natural activity of students by, for example, having them demonstrate and defend their work for their peers. Such prompting helped learners to express the logic of their work by putting their actions into words and symbols. It also enabled other students to "see" possibilities for new flags and questions about flags that they might ask of themselves. The teacher also built on students' actions by having them create flags that represented certain conditions, using such prompts as "Tanya made up a flag that shows one-half minus a sixth and two twenty-fourths. Can you do one like that?"

Katy's and Tina's Flags

THE DISCUSSION THAT FOLLOWS SHOWS THE kind of task adjustment that students made as they worked on this activity. Katy, after unsuccessfully attempting to construct a Canadian flag resembling that in **figure 1**—a red maple leaf on a white strip that is flanked by two narrower red strips—settled for two sixth pieces on a half-piece background **(fig. 6)**.

At first little opportunity for much interesting mathematics seemed to emerge from this flag. Yet when Katy and Tina were asked how much background was uncovered, Katy approached the task by mentally subtracting the area of the panels from the area of the background piece to obtain an answer of one-sixth. Tina arrived at the same value more quickly by mentally filling in the blank spot with a

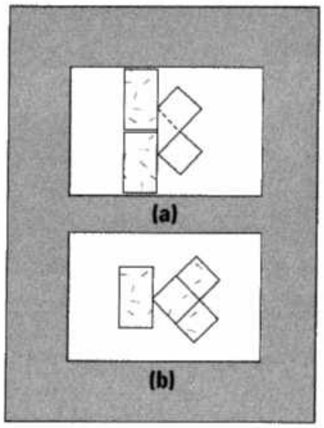


Fig. 7 Tina's flags

sixth piece. Their diverse approaches prompted a discussion of the sorts of activities that might appropriately be considered mathematical. Both agreed that Katy's subtraction method was valid, but Tina was unconvinced that her own approach was "mathematical." Attempting to convince Tina of its validity, Katy offered, "You're kind of estimating that this is about the same thing. That makes it math. You kind of measure in your head." For us, this interaction demonstrates that the value of the learning experience cannot be determined by the teacher's perceptions of complexity or difficulty; rather, the value depends on the students' own experiences, and it is to such experiences that we must attend.

Tina assembled a flag with a capital K in the center of a half piece. The K was made from two twenty-fourths pieces and a folded eighth piece (see **fig. 7a**). When asked how much of the background of this flag was not covered, once again Tina and Katy approached the task differently. Katy attempted to add the covering pieces then subtract, whereas Tina tried to determine which pieces would fill the blank space. Both, however, were frustrated in dealing with the folded piece. Tina reassembled another K using three twenty-fourths pieces (see **fig. 7b**), thus avoiding the need to determine the area of the folded eighth while creating the problem of overlapping pieces.

Once again we saw students negotiating the difficulty of their tasks. Of course, the teacher, if listening to such negotiations, must decide whether to provoke the students to more sophisticated challenges. Still we found repeatedly that in the end, the students' actions determined the appropriate sophistication of the tasks. In this example, Tina attempted to deal with the overlapping pieces by adding their areas but later acknowledged that her answer was incomplete.

It was apparent that these students enjoyed participating in mathematical activities that gave them an opportunity to express themselves aesthetically. When Katy and Tina were asked to generate some questions based on these and other flags, both eagerly developed problems of appropriate complexity—in their words, "hard, but so you can still do it." Students appeared to try to make flags that looked interesting and held mathematical challenges for them. They enjoyed sharing their most interesting and best flags, solving the mathematical problems posed for these flags, and guiding their partners and classmates through their reasoning.

In this activity, Katy and Tina introduced two new ideas with their uses of folded and overlapped pieces. Although not fully pursued by the girls, they were picked up by the teacher as possible curriculum components and were later presented to other students. The result of one such event is discussed subsequently.

Greg's Flags

IT IS OFTEN THOUGHT THAT CONCRETE ACTIVITIES are useful for average and weak students but are not appropriate for strong mathematics students. But even one of the most advanced students, Greg, noted that he had not been bored in the study of fractions—in spite of having demonstrated the prescribed curriculum outcomes in the pretest. In particular, "fraction flags" fulfilled Greg's criteria of a good activity. They were, in his words, "fun and challenging."

The flags gave Greg an opportunity both to develop his calculation skills and to invent situations that required more flexible strategies. Taking the lead from Katy, the teacher posed the question of the area covered by a piece folded at a right angle. Greg was first presented with a folded sixth **(fig. 8),** and his strategy was to draw pencil lines along the overlapping edges of the piece. When he unfolded it, a pair of parallel lines appeared on the piece, the area between which he estimated to be one-twelfth by comparing it with the appropriate fraction piece from the kit. Folding it again, apparently to ensure that half of this twelfth area was covered by the fold, Greg responded that the area covered by the folded piece was three twenty-fourths.

Asked to elaborate, he reasoned, "The space between the lines is about one-twelfth, and half of it gets covered. So the area is the whole sixth minus a half of a twelfth. That's three twenty-fourths." Although he did not express it symbolically, Greg's calculation might be expressed as

$$\frac{1}{6}-\left(\frac{1}{2}\times\frac{1}{12}\right)=\frac{3}{24},$$

again demonstrating the potential sophistication of students' actions in this exploration. Notice that

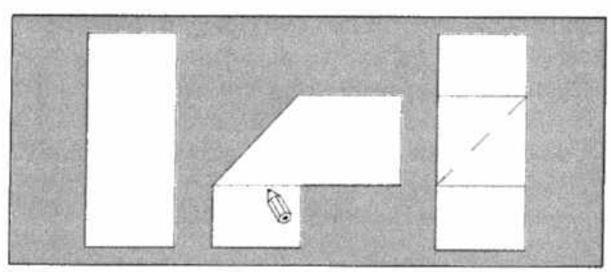


Fig. 8 A folded sixth piece

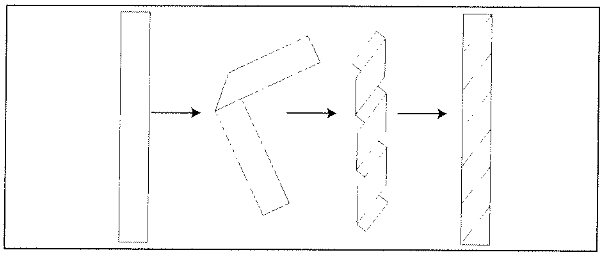


Fig. 9 Greg's folded eighth piece

even this mathematically advanced student made use of the physical materials to represent and check his own thinking while demonstrating more sophisticated understandings.

Later in the discussion, when Greg was creating one of his own flags, he folded an eighth piece in a similar way but, for some reason, selected an alternative method for calculating the area under the fold. His strategy this time was to fold the paper strip in a twisted pattern **(fig. 9).** Untwisting, he proceeded to count the sections created by the spiral fold. His explanation follows:

> There are five parallelograms and two triangles. That makes six shapes the same size, so each is a forty-eighth. . . . One-half of the one forty-eighth is covered. So that's one ninety-sixth. [O]ne-eighth is twelve ninety-sixths, so the [original folded eighth] piece is eleven ninety-sixths.

Symbolically, his action can be expressed as

$$\frac{1}{8}-\frac{1}{2}\left(\frac{1}{6}\times\frac{1}{8}\right)=\frac{11}{96}.$$

Greg's second method bears a strong resemblance to the folding activities that were used to introduce the unit, indicating that when faced with a difficult problem, he reverted to a more primitive strategy to find a solution. It is also interesting to note the complexity of the operations being performed. Although no operation was represented symbolically during the interaction, it is clear that the level of conceptual development fell well beyond curriculum requirements.

Many other "fraction flags" problems were posed to and by the students. One issue in particular—that of doubling the sizes of given flags—proved particularly intriguing. The reader is urged to investigate the problem by selecting one of the flags that has been presented and, *using the pieces supplied in the kit,* creating a flag that is its double. Better still, try the activity with some students who are comfortable with the fraction-flags context. In our unit, this question unexpectedly prompted discussions not just of fraction operations but of proportion, area, and alternative interpretations of "doubling." Once again, attending to the work and concerns of the students allowed for both interesting connections with other mathematical ideas and extensions of the rational-number curriculum.

Closing Remarks

THE "FRACTION FLAGS" ACTIVITY EMBODIES A number of features that we believe are important to effective mathematics teaching and learning. To begin, it is a mathematically rich activity that invites exploration and conjecture while offering opportunities for personal and aesthetic expression. Mathematics learning is viewed as interpersonal, and the "fraction flags" setting allows students to interact with one another, with the teacher, and with the objects in their world. The resulting conversations enable the teacher, in response, to adapt learning activities that are in harmony with students' knowledge and interests. Indeed, "fraction flags" emerged in just this way.

Further, as was repeatedly demonstrated in this activity, students would create problems for themselves that were appropriate to their own levels of understanding. Thus, rather than burden the teacher with the task of simultaneously monitoring and personalizing the mathematics of thirty learners, such activities might help to address effectively one of the most persistent and frustrating problems faced by the teacher: setting tasks for students whose levels of conceptual development vary widely.

Finally, the sorts of problems and activities that emerged with "fraction flags" offered not only opportunities for exploration but also extensive practice of various fraction operations. During the unit it was not unusual for students to generate spontaneously and solve dozens of arithmetic questions without persistent prompting or constant surveillance by the teacher.

These understandings and insights point to different perspectives on both teaching and mathematics. Teaching, as we have attested, involves not simply helping students to learn but, more fundamentally, learning from the learners. Mathematics, similarly, is not a collection of facts to be mastered or objectives to be met but, in Greg's words, "a way of asking questions about the world."

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